ALGEBRA.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

41. Proposed by A. H. BELL, Hillsboro, Illinois.

In a right-angled triangle there are given, the bisectors of the acute angles, Required the triangle.

I. Solution by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland, and D. G. DORRANCE, Jr., Camden, N. Y.

Represent the half-angles by α and $(45-\alpha)$; then easily is deduced

$$\tan 2\alpha = n\cos(45-\alpha)/m\cos\alpha \ldots .(1).$$

$$\therefore\ \tan^3\alpha+\tan^2\alpha+[(2m\surd2-n)/n]\tan\alpha-n=0\ldots .(2);$$

that is, $(\tan\alpha-Q_1)(\tan\alpha-Q_2)(\tan\alpha-Q_3)=0\ldots .(3)$.

Hence three sets of values of the sides of the required right triangle are possible. *Numeralizing* m and n in (2), we deduce Q_1, Q_2, and Q_3 from (3); then α is known. Consequently the three sides, $b=m\cos\alpha$, $p=n\cos(45-\alpha)$, and $h=m\cos\alpha\sec2\alpha$, are known.

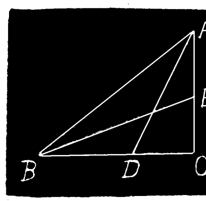
II. Solution by G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.

Let $BC=x$, $AC=nx$, $AB=mx$, $AD=a$, $BE=b$.

Then $m^2-n^2=1\ldots .(1)$.

$$ma+na=2mnx\cos\tfrac12A=\frac{2mn^2x^2}{a}\ldots .(2),$$

$$b+mb=2mx\cos\tfrac12B=\frac{2mx^2}{b}\ldots .(3).$$



$$(2)\div(3)\text{ gives}\frac{(m+n)a^2}{(m+1)b^2}=n^2\ldots .(4).$$

Eliminating n between (1) and (4),

$$m^6+2m^5-\left(1+\frac{2a^2}{b^2}\right)m^4-\left(4+\frac{2a^2}{b^2}\right)m^3-\left(1-\frac{2a^2}{b^2}\right)m^2+2\left(1+\frac{a^2}{b^2}\right)m+\frac{a^4}{b^4}+1=0.$$

Let $\frac{a}{b}=u$. Then $m^6+2m^5-(1+2u^2)m^4-(4+2u^2)m^3-(1-2u^2)m^2+2(1+u^2)m+u^4+1=0$.

To give a more complete solution of this equation might be interesting but well-nigh impossible unless we use numerical results. Such a solution, however, is as unsatisfactory as the problem itself.

Let $u^2=\frac{128}{81}$. Then $3m=5$ or $m=\frac{5}{3}$, $n=\frac{4}{3}$, $x=3$.

$\therefore\quad mx=5,\ nx=4,\ x=3$. Let $u^2=4$. Then $m=1.332$, $n=.8799$, $x=.936-$. $\therefore\quad mx=1.246$, $nx=.8235$, $x=.936$, when $a=2$, $b=1$.

Let $a=40$, $b=50$, $u^2=\frac{16}{25}$, Then $m=1.2532$, $n=.7553$, $x=47.40,2+$.

$\therefore\quad mx=59.4107+,\ nx=35.8067+,\ x=47.4072+$.

Let $a=b=c$, then $u^2=1$, $m=\sqrt{2}$, $n=1$, $x=\frac{c}{2}\sqrt{(2+\sqrt{2})}$.

$$\therefore\quad mx=\frac{c}{\sqrt{2}}\sqrt{(2+\sqrt{2})},\ nx=x=\frac{c}{2}\sqrt{(2+\sqrt{2})}.$$

III. **Solution by B F. BURLESON, Oneida Castle, New York**

Let ABC be the triangle, right angled at C. Put $AD=a=40$, and $BE=b=50$, the lines bisecting the acute angles A and B. Put $x=AB$, $y=AC$, and $z=BC$. Put $\phi+\theta=$the$\angle CAD$ and $\phi-\theta=$the $\angle CBE$. We have, by Trigonometry,

$$x=b\cos(\phi-\theta),\ \ldots.(1),$$
$$y=a\cos(\phi+\theta),\ \ldots.(2),$$
$$y=z\tan(2\phi-2\theta)\ \ldots.(3),$$

Eliminating from (1), (2), and (3), we obtain by development

$(b+b\tan\phi\tan\theta)\left(\dfrac{1-\tan^2\theta-2\tan\theta}{1-\tan^2\theta+2\tan\theta}\right)=0\ldots.(4)$. This is true because $\phi=22\frac{1}{2}^\circ$.

Clearing (4) of fractions, resolving factors, and substituting for $\tan\phi=22\frac{1}{2}^\circ$ its equal $\sqrt{2}-1$, observing that $\cot\phi=\sqrt{2}+1$, we get $(b+a)\tan^3\theta+\{(b-a)(\sqrt{2}+1)+[2(b-a)]\}\tan^2\theta+\{2(b+a)(\sqrt{2}+1)-(b+a)\}\tan\theta=(b-a)(\sqrt{2}-1)\ldots.(5)$. Dividing (5) by $b+a$ and substituting the numerical values of a and b, we get
$\tan^3\theta-.490468\tan^2\theta+3.828427125\tan\theta=.268245951375$. Hence, by Horner's Method of Detached Coefficients, $\tan\theta=.0693633$, and the auxiliary angle $\theta=3^\circ58'4\frac{7}{8}''$. By substituting in (1) and (2), we determine that $y=35.807338$ and $z=47.407325$. $\therefore\quad x=\sqrt{(y^2+z^2)}=59.410604$.

This problem was also solved by *A. H. Bell*, *J. F. W. Scheffer*, and *H. C. Wilkes*.

42. **Proposed by ALEXANDER MACFARLANE, A. M., D.Sc , LL. D., Cornell University, Ithaca, New York.**

There are p electors and q candidates for r seats. Each elector has r votes, and he may distribute them as he pleases among the candidates. Find in how many different ways the voting may result, that is, the number of possible states of the poll.

Solution by G. B. M. ZERR, Staunton, Virginia, and F. P .MATZ, New Windsor, Maryland.

The number of different ways of voting for r seats out of q candidates, when each elector casts r votes for r different persons, is

$$n=\frac{q(q-1)(q-2)(q-3)\ldots.(q-r+1)}{1.2.3.4\ldots.r}.$$